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SUMMARY

1. PURPOSE. To provide security and policy review on the document at Tab 1 prior to release to the public.

2. BACKGROUND.
Authors: Dr. Kurt Herzinger and Lt. Col. Trae Holcomb

Title: Perfect Bricks of Every Size

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3. DISCUSSION.

4. VIEWS OF OTHERS.

5. RECOMMENDATION. Approve document for public release. Suitability is based solely on the document being unclassified, not jeopardizing DoD interests, and accurately portraying official policy.

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1 Tabs
1. Journal Article

Perfect Bricks of Every Size

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Abstract. *We answer an open question from a previous investigation related to numerical semigroups. For integers $k, n \geq 2$ we prove the existence of a numerical semigroup S and a relative ideal I such that the size of the minimal generating set for I is k, the size of the minimal generating set for the dual, $S - I$, is n, and the size of the minimal generating for the ideal sum $I + (S - I)$ is nk. Further, we outline a method for proving that S is symmetric and $S + (S - I) = S\backslash\{0\}$. The primary tool in this investigation is the Apery set of S relative to the multiplicity of S.*

Introduction. Let S be a numerical semigroup and let I be a relative ideal of S. Let $S - I$ denote the *dual* of I, that is, $S - I = \{z \in \mathbb{Z} \mid z + I \subseteq S\}$. We use the notation $\mu_S(\cdot)$ to represent the size of the minimal generating set for a relative ideal of S. If the minimal generating set for I is $\{a_1, \ldots, a_k\}$ and the minimal generating set for $S - I$ is $\{b_1, \ldots, b_n\}$, then the set $\{a_i + b_j | 1 \leq i \leq k \text{ and } 1 \leq j \leq n\}$ is a generating set for the ideal sum $I + (S - I)$ but it may not be a minimal generating set. Thus, the inequality $\mu_S(I)\mu_S(S-I) \geq \mu_S(I+(S-I))$ always holds. When $\mu_S(I)\mu_S(S-I) = \mu_S(I+(S-I))$, we refer to the pair (S, I) as a $k \times n$ *brick*. When $I + (S - I) = S\backslash\{0\}$, we refer to (S, I) as a $k \times n$ *perfect brick* (see [5], [6], and [7] for background). It is known (see [5]) that if $\mu_S(I) = 1$, then $\mu_S(S - I) = 1$ and (S, I) is a trivial brick. Because of this, we always assume that $\mu_S(I) \geq 2$.

From [6] we know there exists an infinite family of 2×2 perfect bricks. At the end of [7] it is conjectured that there exist $k \times n$ perfect bricks for all $k, n \geq 2$. In this paper we confirm this conjecture by constructing a family of perfect bricks of every size $k \times n$ for $k, n \geq 2$. We also compute the Frobenius number for each member of this inifinite family and outline a proof that each one is symmetric.

In Section 1 we introduce the needed notation and define the semigroup S and relative ideal I for given values of $k = \mu_S(I)$ and $n = \mu_S(S - I)$. The multiplicity of S, $m(S)$, is the smallest positive element of S. The analysis in Section 2 focuses on the Apery set of S relative to $m(S)$, that is, the set $Ap(S, m(S)) = \{s \in S | s - m(S) \notin S\}$. Because this is the only Apery set of S that we will use in this investigation, we will henceforth refer to this set as "the Apery set of S" and denote it by $Ap(S)$. Knowing which elements are in $Ap(S)$ will allow us to find the minimial generating set of $S - I$, thus proving that (S, I) is a $k \times n$ perfect brick. We finish by using the Apery set to argue that S is symmetric. In Section 3 we describe an infinite collection of similar families as well as present a family of $k \times n$ perfect bricks which are not symmetric for $k \geq 2$ and $n \geq 3$.

A basic understanding of the concepts and notation related to numerical semigroups and relative ideals is assumed in this paper. Suggested background reading for numerical semigroups and their connections to commutative algebra include [2], [4] and [10]. The original motivation for the investigation in this paper comes from the study of torsion in tensor products of modules over certain types of rings. The specifics of the relationship between this topic and numerical semigroups is detailed in [5]. Details concerning the investigation of torsion in tensor products can be found in [1], [3], [8] and [9].

1. Background, Definitions and Notation

(1.1) Notation: The following notation will be used throughout this investigation.

Let k and n be integers such that $k, n \geq 2$.

(i) For $i = 0, 1, \ldots, n-1$, let $a_i = 3(n+i) - 1$.

(ii) For integers $z_1 < z_2$, let $[z_1, z_2]$ denote the set $\{t \in \mathbb{Z} | z_1 \leq t \leq z_2\}$.

(iii) For $i = 0, 1, \ldots, n-1$, let
$C_i = [(2k-1)a_i, (2k-1)a_i + (k-1)]$.

(iv) For positive integers $s_1, s_2, \ldots, s_p$ with $g.c.d.(s_1, s_2, \ldots, s_p) = 1$, let $S = < s_1, s_2, \ldots, s_p >$ represent the numerical semigroup with minimal generating set $\{s_1, s_2, \ldots, s_p\}$. That is, $S = \{x_1 s_1 + x_2 s_2 + \cdots + x_p s_p\}$ where $x_1, x_2, \ldots, x_p$ are non-negative integers.

For the remaining items, let S be a numerical semigroup.

(v) For integers $b_1, b_2, \ldots, b_m$, let $I = (b_1, b_2, \ldots, b_m)$ represent the relative ideal of S with generating set $\{b_1, b_2, \ldots, b_m\}$. That is, $I = \{b_1, b_2, \ldots, b_m\} + S$.

(vi) For a relative ideal I, we represent the dual of I in S by $S - I$. That is, $S - I = \{z \in \mathbb{Z} | z + I \subseteq S\}$.

(vii) It is well known that given a relative ideal I of S, its minimal generating set is unique. We will denote the number of elements in the minimal generating set for I by $\mu_S(I)$.

We can now define the numerical semigroup S and relative ideal I which form a $k \times n$ perfect brick.

(1.2) Definition: For a given pair of integers k and n with $k, n \geq 2$, define

$$S = < C_0 \cup C_1 \cup \ldots \cup C_{n-1} >$$

and

$$I = (0, 1, \ldots, k-1).$$

(1.3) Note: (a) Because it will come up several times in this investigation, we note that $a_i - a_0 = 3i$ for $i = 0, 1, \ldots, n-1$. Also note that

$$m(S) = (2k-1)a_0 = (2k-1)(3n-1).$$

(b) The set $C_0 \cup C_1 \cup \ldots \cup C_{n-1}$ is the minimal generating set for S. To see this note that the largest element in this set is $(2k-1)a_{n-1} + (k-1)$. Now,

$$\begin{aligned} 2m(S) - ((2k-1)a_{n-1} + (k-1)) & \\ &= 2(2k-1)a_0 - ((2k-1)a_{n-1} + (k-1)) \\ &= 2(2k-1)(3n-1) - (2k-1)(6n-4) - (k+1) \\ &= (2k-1)(6n-2-6n+4) - (k+1) \\ &= (2k-1)(2) - (k+1) \\ &= 3k-1 \\ &> 0. \end{aligned}$$

We conclude that all elements of $C_0 \cup C_1 \cup \ldots \cup C_{n-1}$ are less than $2m(S)$.

(1.4) Example: Let $k = 3$ and $n = 4$. Then

$$S = <55, 56, 57, 70, 71, 72, 85, 86, 87, 100, 101, 102> \text{ and } I = (0, 1, 2).$$

It is easy to check that $S - I = (55, 70, 85, 100)$, $\mu_S(I + (S - I)) = 12$, and $I + (S - I) = S \backslash \{0\}$. Thus (S, I) forms a 3×4 perfect brick.

For a given k and n, the numerical semigroup S has a minimal generating set consisting of n intervals of consecutive integers consisting of k integers each. Further, I has $[0, k-1]$ as its minimal generating set. In general, a numerical semigroup generated by intervals of consecutive integers and a relative ideal generated by an interval do not form a brick. As we will see in the analysis that follows, the additional conditions we have imposed on S and I ensure that the pair (S, I) always forms a $k \times n$ perfect brick.

For a specific k and n, it is easy to check that the pair defined in (1.3) is a $k \times n$ perfect brick. The challenge comes in developing a general proof that is valid for all values of k and n. The approach we will take involves establishing which elements of S are in the Apery set of S. Recall that for this investigation, we define the Apery set of S to be

$$Ap(S) = \{s \in S | s - m(S) \notin S\}.$$

Equivalently, $Ap(S)$ consists of the smallest elements of S from each congruence class modulo $m(S)$. Knowing the elements of $Ap(S)$ will benefit us in two ways.

1. By the way S and I are defined, we know

$$(2k-1)a_0, (2k-1)a_1, \ldots, (2k-1)a_{n-1}$$

are members of the minimal generating set for $S - I$. To see this note that $(2k-1)a_i + j \in S$ for $0 \leq i < n$ and $0 \leq j \leq k-1$. By definition, $(2k-1)a_i \in S - I$. Further, recall from (1.3) that $(2k-1)a_{n-1} < 2m(S)$. As a result, all of these elements are members of the minimal generating set for $S - I$.

Now, if $x \in S \backslash Ap(S)$, then we know that $x - m(S) \in S$. As a result, x is not a member of the minimal generating set for $S - I$ since $m(S) \in S - I$. We conclude that if x is a member of the minimal generating set for $S - I$, then $x \in Ap(S)$. Knowing the elements of $Ap(S)$ allows us to narrow the search for members of this generating set. In the end we will discover that the minimal generating set for $S - I$ contains no elements other than $(2k-1)a_0, (2k-1)a_1, \ldots, (2k-1)a_{n-1}$ and hence $\mu_S(S - I) = n$.

2. We will outline a method by which we can confirm that S is symmetric. It is known (see [11]) that S is symmetric if and only if $Max(Ap(S)) - x \in Ap(S)$ for all $x \in Ap(S)$.

2. Analysis of $Ap(S)$ and the Minimal Generating Set of $S - I$

We now establish which elements of S are in $Ap(S)$ and determine which ones are elements of the minimal generating set of $S-I$. The process involves looking at how a given element of S can be expressed with respect to the minimal generating set.

Let $T_0 = \{0\}$ and for $j \geq 1$ let $T_j = \{x \in S | x = g_1 + g_2 + \cdots + g_j\}$ where $g_1, g_2, \ldots, g_j$ are elements of the minimal generating set of S. Note that

$$T_j \subset \bigcup_{0 \leq i_1, \ldots, i_j \leq n-1} C_{i_1} + \cdots + C_{i_j}.$$

We will examine T_0, T_1, T_2, T_3, T_4 and discover that $Ap(S) \subseteq T_0 \cup T_1 \cup T_2 \cup T_3 \cup T_4$.

To help us with our bookkeeping we will partition the congruence classes modulo $m(S)$ as follows:

$$P_i = i(2k-1) + [0, 2k-2]$$

for $0 \leq i \leq 3n-2$. That is, each P_i consists of $2k-1$ consecutive integers and $\bigcup P_i = [0, m(S)-1]$.

Analysis of T_0 and T_1

It is clear that $T_1 = C_0 \cup C_1 \cup \ldots \cup C_{n-1}$. Thus, a typical element of T_1 looks like $x = (2k-1)a_i + l$ where $0 \leq i \leq n-1$ and $0 \leq l \leq k-1$. By (1.3) we know $x - m(S)$ is the congruence class representative for x modulo $m(S)$. Further,

$$\begin{aligned} x - m(S) &= (2k-1)a_i + l - (2k-1)a_0 \\ &= (2k-1)(a_i - a_0) + l \\ &= (2k-1)(3i) + l. \end{aligned}$$

As i varies from 0 to $n-1$ and l varies from 0 to $k-1$, we see that x corresponds to the congruence classes

$$3i(2k-1) + [0, k-1] \subseteq P_{3i}.$$

As a result, the elements of $Ap(S)$ along with the obvious element, namely 0, are

$$[1, k-1] + m(S)$$

and

$$3i(2k-1) + [0, k-1] + m(S),$$

where $1 \leq i \leq n-1$.

Because the integers

$$(2k-1)a_0 + k, (2k-1)a_1 + k, \ldots (2k-1)a_{n-1} + k$$

are not in S, it is clear that the only elements of $T_0 \cup T_1$ that are in $S - I$ are

$$(2k-1)a_0, (2k-1)a_1, \ldots, (2k-1)a_{n-1}.$$

As we saw above, these elements are in the minimal generating set for $S - I$.

Finally note that $T_0 \cup T_1$ consists of the elements of S that are less than $2m(S)$.

Before proceeding with the analysis of T_2, we offer the following lemmas that will be useful in the analysis that follows.

(2.1) Lemma: Let $x \in S$ such that $x = (2k-1)a_{i_1} + \cdots + (2k-1)a_{i_j} + l$ where $0 \leq i_1, \ldots, i_j \leq n-1$ and $0 \leq l \leq j(k-1)$. Let $m \geq 0$. If $i_1 + \cdots + i_j \geq mn$, then $x \geq (j+m)m(S)$.

Proof: Assume $i_1 + \cdots + i_j \geq mn$. Then

$$\begin{aligned}
x - (j+m)m(S) &= (2k-1)(a_{i_1} + \cdots + a_{i_j}) + l - (j+m)m(S) \\
&= (2k-1)(a_{i_1} + \cdots + a_{i_j}) + l - (j+m)(2k-1)a_0 \\
&= (2k-1)(a_{i_1} + \cdots + a_{i_j} - (j+m)a_0) + l \\
&= (2k-1)(3i_1 + \cdots + 3i_j - ma_0) + l \\
&= (2k-1)(3(i_1 + \cdots + i_j) - m(3n-1)) + l \\
&= (2k-1)(3(i_1 + \cdots + i_j - mn) + m) + l \\
&\geq 0.
\end{aligned}$$

Therefore, $x \geq (j+m)m(S)$.

(2.2) Lemma: Let $x \in S$ such that $x = (2k-1)a_{i_1} + (2k-1)a_{i_2} + \cdots + (2k-1)a_{i_j} + l$ where $0 \leq i_1, \ldots, i_j \leq n-1$ and $0 \leq l \leq (j-1)(k-1)$. Then x is not a member of the minimal generating set for $S - I$.

Proof: Write l as $l = l_1 + l_2 + \cdots + l_{j-1}$ where $0 \leq l_1, l_2, \ldots, l_{j-1} \leq k-1$. Then,

$$x = ((2k-1)a_{i_1} + l_1) + ((2k-1)a_{i_2} + l_2) + \cdots + ((2k-1)a_{i_{j-1}} + l_{j-1}) + (2k-1)a_{i_j}.$$

Now, $(2k-1)a_{i_m} + l_m \in C_{i_m}$ for $m = 1, 2, \ldots, j-1$, so $x = (2k-1)a_{i_j} + s$ for some $s \in S$. Since $(2k-1)a_{i_j} \in S - I$, we see that x is not a member of the minimal generating set for $S - I$.

Analysis of T_2

By definition we see that

$$T_2 = \bigcup_{0 \leq i,j \leq n-1} C_i + C_j.$$

Therefore, a typical element of T_2 looks like

$$x = (2k-1)a_{i_1} + l_{i_1} + (2k-1)a_{i_2} + l_{i_2},$$

where $0 \leq i_1, i_2 \leq n-1$ and $0 \leq l_{i_1}, l_{i_2} \leq k-1$. Rewriting we have

$$x = (2k-1)(a_{i_1} + a_{i_2}) + l,$$

where $0 \leq l \leq 2k-2$. We want to reduce this element modulo $m(S) = (2k-1)a_0$ to discover to which congruence class modulo $m(S)$ it corresponds. Since $x \geq 2m(S)$, we start by substracting $2m(S)$ to obtain

$$\begin{aligned} x - 2m(S) &= (2k-1)(a_{i_1} + a_{i_2}) + l - 2(2k-1)a_0 \\ &= (2k-1)(a_{i_1} + a_{i_1} - 2a_0) + l \\ &= (2k-1)(3i_1 + 3i_2) + l \\ &= (2k-1)(3(i_1 + i_2)) + l. \end{aligned}$$

We now examine two cases based upon the value of $i_1 + i_2$. In case (1) we let $0 \leq i_1 + i_2 \leq n-1$, and in case (2) we let $n \leq i_1 + i_2 \leq 2n-2$.

In case (1) we have

$$\begin{aligned} (2k-1)(3(i_1+i_2)) + l &\leq (2k-1)(3(n-1)) + l \\ &\leq (2k-1)(3n-3) + (2k-2) \\ &= (2k-1)(3n-1-2) + (2k-2) \\ &= (2k-1)(3n-1) - 2(2k-1) + (2k-2) \\ &= m(S) - 2k \\ &< m(S). \end{aligned}$$

Thus x is in the congruence class corresponding to $(2k-1)(3(i_1+i_2)) + l$. As $i_1 + i_2$ varies from 0 to $n-1$ and l varies from 0 to $2k-2$, we see that x corresponds to the congruence classes

$$3(i_1+i_2)(2k-1) + [0, 2k-2] = P_{3(i_1+i_2)}.$$

Recalling the analysis of T_0 and T_1, we see that

$$3(i_1+i_2)(2k-1) + [k, 2k-2] + 2m(S)$$

are elements of $Ap(S)$ where $0 \leq i_1 + i_2 \leq n-1$.

In case (2) we know $x \geq 3m(S)$ by (2.1), so we have to subtract $3m(S)$ to find the congruence class to which x corresponds. Specifically,

$$\begin{aligned} x - 3m(S) &= (2k-1)(a_{i_1} + a_{i_2}) + l - 3(2k-1)a_0 \\ &= (2k-1)(a_{i_1} + a_{i_2} - 3a_0) + l \\ &= (2k-1)(3i_1 + 3i_2 - (3n-1)) + l \\ &= (2k-1)(3(i_1 + i_2 - n) + 1) + l. \end{aligned}$$

Furthermore,

$$\begin{aligned} (2k-1)(3(i_1+i_2-n)+1)+l &\leq (2k-1)(3(2n-2-n)+1)+l \\ &\leq (2k-1)(3(n-2)+1)+(2k-2) \\ &= (2k-1)(3n-5)+(2k-2) \\ &= (2k-1)(3n-1-4)+(2k-2) \\ &= m(S) - 6k + 2 \\ &< m(S). \end{aligned}$$

Thus we know x is in the congruence class corresponding to

$$(2k-1)(3(i_1+i_2-n)+1)+l.$$

As $i_1 + i_2$ varies from n to $2n-2$ and l varies from 0 to $2k-2$, we see that x corresponds to the congruence classes

$$(3(i_1+i_2-n)+1)(2k-1) + [0, 2k-2] = P_{3(i_1+i_2-n)+1}.$$

We conclude that

$$(3(i_1+i_2-n)+1)(2k-1) + [0, 2k-2] + 3m(S)$$

are elements of $Ap(S)$ where $n \leq i_1 + i_2 \leq 2n-2$.

We now address the question of which elements of $T_2 \cap Ap(S)$ are in the minimal generating set of $S - I$. Recall that we need only examine the elements of $Ap(S)$.

For $0 \leq i_1 + i_2 \leq n-1$ we have

$$3(i_1+i_2)(2k-1) + [k, 2k-2] + 2m(S) \subseteq Ap(S).$$

None of these elements are in $S - I$ because the next largest integer is $3(i_1+i_2)(2k-1)+(2k-1)+2m(S) = (3(i_1+i_2)+1)(2k-1)+2m(S)$ which is not in S. To see this note that $(3(i_1+i_2)+1)(2k-1)+2m(S)$ is smaller than $3m(S)$ and corresponds to the congruence class $(3(i_1+i_2)+1)(2k-1) \in$

$P_{3(i_1+i_2)+1}$. For $0 \le i_1+i_2 \le n-2$, the analysis above shows that the smallest representative of this congruence class in S is greater than $3m(S)$. For $i_1+i_2 = n-1$, this element corresponds to a congruence class in P_{3n-2}. As we will see later, the smallest representative of this congruence class in S is also greater than $3m(S)$. Thus, $(3(i_1+i_2)+1)(2k-1)+2m(S) \notin S$ and we conclude that none of the elements in $3(i_1+i_2)(2k-1)+[k, 2k-2]+2m(S)$ are in $S-I$.

Next note that for $n \le i_1+i_2 \le 2n-2$ we have

$$(3(i_1+i_2-n)+1)(2k-1)+[0, 2k-2]+3m(S) \subseteq Ap(S).$$

We first consider the elements

$$(3(i_1+i_2-n)+1)(2k-1)+[k, 2k-2]+3m(S).$$

None of these elements are in $S-I$ because the next largest integer is $(3(i_1+i_2-n)+1)(2k-1)+(2k-1)+3m(S) = (3(i_1+i_2-n)+2)(2k-1)+3m(S)$ which is not in S. To see this note that $(3(i_1+i_2-n)+2)(2k-1)+3m(S)$ is smaller than $4m(S)$ and corresponds to the congruence class $(3(i_1+i_2-n)+2)(2k-1) \in P_{3(i_1+i_2-n)+2}$. But, as we will see in the analysis of T_3 and T_4, the smallest representative of this congruence class in S is greater than $4m(S)$. Thus, $(3(i_1+i_2-n)+2)(2k-1)+3m(S) \notin S$ and we conclude that none of the elements in $(3(i_1+i_2-n)+1)(2k-1)+[0, 2k-2]+3m(S)$ are in $S-I$.

The only elements of $Ap(S)$ in T_2 left to consider are

$$(3(i_1+i_2-n)+1)(2k-1)+[0, k-1]+3m(S).$$

It is clear that these elements are in $S-I$. Let x be an element of S in this interval. Recall that x can be written in the form $x = (2k-1)(a_{i_1}+a_{i_2})+l$ where $n \le i_1+i_2 \le 2n-2$ and $0 \le l \le k-1$. From (2.2), we conclude that x is not a member of the minimal generating for $S-I$ and therefore T_2 contains no elements that are in the minimal generating set for $S-I$.

Before moving on to the analysis of T_3 and T_4, it is helpful to recap what we know. We have identified all of the elements of $Ap(S)$ in the interval $[m(S), 2m(S)]$ and in the interval $[2m(S), 3m(S)]$. These elements correspond precisely to the congruence classes in $P_0, P_3, P_6, \ldots, P_{3n-3}$. We have also established <u>some</u> of the elements of $Ap(S)$ in the interval

$[3m(S), 4m(S)]$. These elements correspond precisely to the congruence classes in $P_1, P_4, \ldots, P_{3n-5}$. At this point we have not encountered any elements of S that correspond to the congruence classes in $P_2, P_5, P_8, \ldots, P_{3n-4}$. Further, we have not encountered any elements of S that correspond to the congruence classes in P_{3n-2}.

Analysis of T_3

By definition we see that

$$T_3 \subseteq \bigcup_{0 \le i_1, i_2, i_3 \le n-1} C_{i_1} + C_{i_2} + C_{i_3}.$$

Therefore, a typical element of T_3 looks like

$$x = (2k-1)a_{i_1} + l_1 + (2k-1)a_{i_2} + l_2 + (2k-1)a_{i_3} + l_3,$$

where $0 \le i_1, i_2, i_3 \le n-1$ and $0 \le l_1, l_2, l_3 \le k-1$. Rewriting we have

$$x = (2k-1)(a_{i_1} + a_{i_2} + a_{i_3}) + l,$$

where $0 \le l \le 3k-3$. We reduce x modulo $m(S)$ to discover to which congruence class it corresponds. Since $x \ge 3m(S)$ we start by subtracting $3m(S)$ from x. As in the analysis of T_2, one can show that

$$x - 3m(S) = (2k-1)(3(i_1 + i_2 + i_3)) + l.$$

We consider the following cases based upon the value of $i_1 + i_2 + i_3$:

$$\begin{array}{rr}
(1) & 0 \le i_1 + i_2 + i_3 \le n-1 \\
(2) & n \le i_1 + i_2 + i_3 \le 2n-2 \\
(3) & i_1 + i_2 + i_3 = 2n-1 \\
(4) & 2n \le i_1 + i_2 + i_3 \le 3n-3.
\end{array}$$

For case (1), one can show that $(2k-1)(3(i_1+i_2+i_3))+l < m(S)$. Thus we know that x is in the congruence class corresponding to $(2k-1)(3(i_1 + i_2 + i_3)) + l$. As $i_1 + i_2 + i_3$ varies from 0 to $n-1$ and l varies from 0 to $3k-3$, we see that x corresponds to the congruence classes

$$3(i_1 + i_2 + i_3)(2k-1) + ([0, 2k-2] \cup [2k-1, (2k-1) + (k-2)]),$$

which are contained in $P_{3(i_1+i_2+i_3)} \cup P_{3(i_1+i_2+i_3)+1}$.

Reviewing the analysis of T_0, T_1 and T_2, we see that the congruence classes in the above collection that we have not yet encountered are

$$(3n-2)(2k-1) + [0, k-2] \subseteq P_{3n-2}.$$

That is,

$$(3n-2)(2k-1) + [0, k-2] + 3m(S) \subseteq Ap(S).$$

For cases (2), (3), and (4), we know $x \geq 4m(S)$ by (2.1). One can show that $x - 4m(S) = (2k-1)(3(i_1+i_2+i_3-n)+1) + l$.

For case (2), we have $(2k-1)(3(i_1+i_2+i_3-n)+1) + l < m(S)$. As $i_1+i_2+i_3$ varies from n to $2n-2$ and l varies from 0 to $3k-3$, we see that x corresponds to the congruence classes

$$(3(i_1+i_2+i_3-n)+1)(2k-1) + ([0, 2k-2] \cup [2k-1, (2k-1)+(k-2)]),$$

which are contained in $P_{3(i_1+i_2+i_3-n)+1} \cup P_{3(i_1+i_2+i_3-n)+2}$.

We see that the congruence classes that we have not yet encountered are

$$(3(i_1+i_2+i_3-n)+1)(2k-1) + [2k-1, (2k-1)+(k-2)] \subseteq P_{3(i_1+i_2+i_3-n)+2}.$$

We conclude that

$$(3(i_1+i_2+i_3-n)+1)(2k-1) + [2k-1, (2k-1)+(k-2)] + 4m(S)$$

are elements of $Ap(S)$, where $n \leq i_1+i_2+i_3 \leq 2n-2$.

Case (3) separates into two subcases based upon the value of l: (a) $0 \leq l \leq 2k-2$ and (b) $2k-1 \leq l \leq 3k-3$.

If $i_1+i_2+i_3 = 2n-1$ and $0 \leq l \leq 2k-2$, then $x - 4m(S) < m(S)$ and x is in the congruence class corresponding to $(2k-1)(3n-2)+l$. As l varies from 0 to $2k-2$, we see that x corresponds to the congruence classes

$$(3n-2)(2k-1) + [0, 2k-2] = P_{3n-2}.$$

The congruence classes in this collection that we have not yet encountered are

$$(3n-2)(2k-1)+[k-1,2k-2] \subseteq P_{3n-2}.$$

That is,

$$(3n-2)(2k-1)+[k-1,2k-2]+4m(S) \subseteq Ap(S).$$

If $i_1+i_2+i_3=2n-1$ and $2k-1 \leq l \leq 3k-3$, one can show that $x-5m(S)<m(S) \leq x-4m(S)$ so that x is in the congruence class corresponding to $(2k-1)(-1)+l$. As l varies from $2k-1$ to $3k-3$, we see that x corresponds to the congruence classes

$$[0,k-2] \subset P_0.$$

Thus, we know that there are elements in the interval $[m(S),2m(S)]$ that correspond to these congruence classes.

In case (4), we know $x \geq 5m(S)$ by (2.1). One can show that $x-5m(S)=(2k-1)(3(i_1+i_2+i_3-2n)+2)+l<m(S)$ so that x is in the congruence class corresponding to $(2k-1)(3(i_1+i_2+i_3-2n)+2)+l$. As $i_1+i_2+i_3$ varies from $2n$ to $3n-3$ and l varies from 0 to $3k-3$, we see that x corresponds to the congruence classes

$$(3(i_1+i_2+i_3-2n)+2)(2k-1)+([0,2k-2]\cup[2k-1,(2k-1)+(k-2)]),$$

which are contained in $P_{3(i_1+i_2+i_3-2n)+2} \cup P_{3(i_1+i_2+i_3-2n)+3}$.

We see that the congruence classes in the above collection that we have not yet encountered are

$$(3(i_1+i_2+i_3-2n)+2)(2k-1)+[k-1,2k-2] \subseteq P_{3(i_1+i_2+i_3-2n)+2}.$$

Thus, $(3(i_1+i_2+i_3-2n)+2)(2k-1)+[k-1,2k-2]+5m(S)$ are elements of $Ap(S)$, where $2n \leq i_1+i_2+i_3 \leq 3n-3$.

We now review the elements of $T_3 \cap Ap(S)$ to deteremine which of them are in the minimal generating set for $S-I$.

In the set $[3m(S),4m(S)]$, the elements of $T_3 \cap Ap(S)$ are

$$(3n-2)(2k-1)+[0,k-2]+3m(S).$$

The next largest integer is $(3n-2)(2k-1)+(k-1)+3m(S)$ which is smaller than $4m(S)$ and corresponds to the congruence class $(3n-2)(2k-$

$1) + (k-1)$. But the smallest element of S that corresponds to this congruence class is greater than $4m(S)$. Thus, $(3n-2)(2k-1)+(k-1)+3m(S) \notin S$. We conclude that none of these elements are in $S - I$.

In the set $[4m(S), 5m(S)]$, the elements of $T_3 \cap Ap(S)$ are

$$(3i+2)(2k-1) + [0, k-2] + 4m(S) \text{ for } i = 0, 1, \dots, n-2$$

and

$$(3n-2)(2k-1) + [k-1, 2k-2] + 4m(S).$$

None of the elements in $(3i+2)(2k-1)+[0,k-2]+4m(S)$ are in $S-I$ because the next largest integer is $(3i+2)(2k-1)+(k-1)+4m(S)$ which is smaller than $5m(S)$ and corresponds to the congruence class $(3i+2)(2k-1)+(k-1)$. But the smallest element of S that corresponds to this congruence class is greater than $5m(S)$. Thus, $(3i+2)(2k-1)+(k-1)+4m(S) \notin S$.

Now, the elements in $(3n-2)(2k-1) + [k-1, 2k-2] + 4m(S)$ are in $S - I$ since the next k integers are $C_0 + 4m(S) \subseteq S$. Let x be an element of S in this interval. Recall from case (3) that x can be written in the form

$$x = (2k-1)(a_{i_1} + a_{i_2} + a_{i_3}) + l,$$

where $i_1 + i_2 + i_3 = 2n - 1$ and $k - 1 \leq l \leq 2k - 2$. From (2.2), we conclude that x is not a member of the minimal generating for $S - I$.

In the set $[5m(S), 6m(S)]$, the elements of $Ap(S)$ are $(3i+2)(2k-1) + [k-1, 2k-2] + 5m(S)$ for $i = 0, 1, \dots, n-3$. These elements are in $S - I$. To see this we note that the next k integers are in S since they are larger than their representatives in $Ap(S)$ which are identified in the analysis of T_0 and T_1. Let x be an element of S in this interval. Recall that x can be written in the form $x = (2k-1)(a_{i_1} + a_{i_2} + a_{i_3}) + l$ where $2n \leq i_1 + i_2 + i_3 \leq 3n - 3$ and $k - 1 \leq l \leq 2k - 2$. From (2.2), we conclude that x is not a member of the minimal generating set for $S - I$. Thus, there are no elements of $T_3 \cap Ap(S)$ in the minimal generating set of $S - I$.

To recap, in $T_0 \cup T_1 \cup T_2 \cup T_3$ we have elements of $Ap(S)$ corresponding to every congruence class except those in the interval

$$(3n-4)(2k-1) + [k-1, 2k-2] \subseteq P_{3n-4}.$$

Analysis of T_4

By definition we see that

$$T_4 \subseteq \bigcup_{0 \le i_1, i_2, i_3, i_4 \le n-1} C_{i_1} + C_{i_2} + C_{i_3} + C_{i_4}.$$

Thus, a typical element of T_4 can be written in the form

$$x = (2k-1)a_{i_1} + l_1 + (2k-1)a_{i_2} + l_2 + (2k-1)a_{i_3} + l_3 + (2k-1)a_{i_4} + l_4,$$

where $0 \le i_1, i_2, i_3, i_4 \le n-1$ and $0 \le l_1, l_2, l_3, l_4 \le k-1$. Rewriting x we have

$$x = (2k-1)(a_{i_1} + a_{i_2} + a_{i_3} + a_{i_4}) + l,$$

where $0 \le l \le 4k-4$. We know $x \ge 4m(S)$ by (2.1). One can show that $x - 4m(S) = (2k-1)(3(i_1+i_2+i_3+i_4)) + l$. The cases we need to address to discover the remaining elements in $Ap(S)$ are (1) $0 \le i_1+i_2+i_3+i_4 \le n-1$, (2) $n \le i_1+i_2+i_3+i_4 \le 2n-2$, (3) $i_1+i_2+i_3+i_4 = 2n-1$, and (4) $2n \le i_1+i_2+i_3+i_4 \le 3n-2$.

In case (1), one can show that $(2k-1)(3(i_1+i_2+i_3+i_4)) + l < m(S)$ so that x is in the congruence class corresponding to $(2k-1)(3(i_1+i_2+i_3+i_4)) + l$. As $i_1+i_2+i_3+i_4$ varies from 0 to $n-1$ and l varies from 0 to $4k-4$, we see that x corresponds to the congruence classes

$$3(i_1+i_2+i_3+i_4)(2k-1) + [0, 2k-2] \cup [2k-1, (2k-1)+(2k-3)],$$

which are contained in $P_{3(i_1+i_2+i_3+i_4)} \cup P_{3(i_1+i_2+i_3+i_4)+1}$.

We have already encountered elements in the interval $[0, 4m(S)]$ that correspond to these congruence classes. Thus, none of these elements are in $Ap(S)$.

In case (2), $x \ge 5m(S)$ by (2.1). It is easy to verify that $x - 5m(S) = (2k-1)(3(i_1+i_2+i_3+i_4-n)+1) + l < m(S)$. Thus x is in the congruence class corresponding to $(2k-1)(3(i_1+i_2+i_3+i_4-n)+1)+l$. Now $i_1+i_2+i_3+i_4$ varies from n to $2n-2$ and l varies from 0 to $4k-4$. Note that for $i_1+i_2+i_3+i_4 = n+t$, where $0 \le t \le n-3$, we know that $x - 5m(S)$ corresponds to congruence classes in $P_{3t+1} \cup P_{3t+2}$, all of which we have accounted for in previous analyses. Thus we look at the case $i_1+i_2+i_3+i_4 = 2n-2$. We see that x corresponds to the congruence classes

$$(3n-5)(2k-1)+[0,2k-2]\cup[2k-1,(2k-1)+(2k-3)]\subseteq P_{3n-5}\cup P_{3n-4}.$$

The congruence classes in the above collection that we have not yet encountered are

$$(3n-4)(2k-1)+[k-1,2k-3].$$

Thus we have

$$(3n-4)(2k-1)+[k-1,2k-3]+5m(S)\subseteq Ap(S).$$

The only congruence class not already accounted for is $(3n-4)(2k-1)+(2k-2)$.

In case (3), we see that x corresponds to congruence classes in $P_{3n-2}\cup P_0$ following the same analysis that we conducted for T_3 in case (3) where $i_1+i_2+i_3=2n-1$. Since we have already accounted for all these congruence classes, we see that this case yields nothing new.

In case (4), $x\geq 6m(S)$ by (2.1). One can show that $x-6m(S)=(2k-1)(3(i_1+i_2+i_3+i_4-2n)+2)+l<m(S)$ so that x is in the congruence class corresponding to $(2k-1)(3(i_1+i_2+i_3+i_4-2n)+2)+l$. Now $i_1+i_2+i_3+i_4$ varies from $2n$ to $3n-2$ and l varies from 0 to $4k-4$. In particular we look at the case $i_1+i_2+i_3+i_4=3n-2$. We see that x corresponds to the congruence classes

$$(3n-4)(2k-1)+[0,2k-2]\cup[2k-1,(2k-1)+(2k-3)]\subseteq P_{3n-4}\cup P_{3n-3}.$$

When $l=2k-2$ we see that x corresponds to the congruence class $(3n-4)(2k-1)+(2k-2)$. We conclude that the largest element of $Ap(S)$ is $(3n-4)(2k-1)+(2k-2)+6m(S)$.

To complete the analysis of $S-I$, we look at the elements of $T_4\cap Ap(S)$. In the interval $[5m(S),6m(S)]$, we have the elements

$$(3n-4)(2k-1)+[k-1,2k-3]+5m(S).$$

None of these elements are in $S-I$ because the next integer, $(3n-4)(2k-1)+(2k-2)+5m(S)$, corresponds to the congruence class $(3n-4)(2k-1)+(2k-2)$, and the smallest representative of this congruence class is $(3n-4)(2k-1)+(2k-2)+6m(S)$. Thus, $(3n-4)(2k-1)+(2k-2)+5m(S)\notin$

S.

Finally we consider $(3n-4)(2k-1)+(2k-2)+6m(S)$, which is clearly in $S-I$ since it is the largest element in $Ap(S)$ and the next $k-1$ integers must therefore be in S. Recall that $x=(3n-4)(2k-1)+(2k-2)+6m(S)$ can be written as $x=(2k-1)(a_{i_1}+a_{i_2}+a_{i_3}+a_{i_4})+l$ where $l=2k-2$. By (2.2), we conclude that $(3n-4)(2k-1)+(2k-2)+6m(S)$ is not in the minimal generating set of $S-I$.

The final conclusion of this analysis is that

$$S-I=((2k-1)a_0,(2k-1)a_1,\ldots,(2k-1)a_{n-1}).$$

Thus, $C_0\cup C_1\cup\ldots\cup C_{n-1}$ is a generating set for $I+(S-I)$. By (1.3) we know this generating set is minimal. Consequently, $\mu_S(I+(S-I))=kn=\mu_S(I)\mu_S(S-I)$ and (S,I) is a $k\times n$ perfect brick.

We note in passing that the Frobenius number of S, that is, the largest integer not contained in S, is $Max(Ap(S))-m(S)=(3n-4)(2k-1)+(2k-2)+5m(S)$.

(2.3) Example: Recall that for $k=3$ and $n=4$ we have

$$S=<55,56,57,70,71,72,85,86,87,100,101,102> \text{ and } I=(0,1,2).$$

It is easy to check that the largest element of $Ap(S)$ is $(3n-4)(2k-1)+(2k-2)+6m(S)=374$.

Symmetry

Verifying that S is symmetric is now a simple matter of performing a finite number of subtractions. It is known from [11] (among others) that a numerical semigroup S is symmetric if and only if $max(Ap(S))-x\in Ap(S)$ for all $x\in Ap(S)$. Since we know the elements of $Ap(S)$, we just have to check these subtractions. The details are left to the reader, but we offer the following example of how to do these checks.

(2.4) Example: We know that $2(2k-1)+[k-1,2k-2]+5m(S)\subseteq Ap(S)$. Let $x\in 2(2k-1)+[k-1,2k-2]+5m(S)$. That is, $x=2(2k-1)+l+5m(S)$ where $k-1\leq l\leq 2k-2$. Subtracting from $max(Ap(S))$, we get

$$\begin{aligned}
&max(Ap(S)) - x \\
&\quad = ((3n-4)(2k-1) + (2k-2) + 6m(S)) - (2(2k-1) - l - 5m(S)) \\
&\quad = (3n-6)(2k-1) + (2k-2) - l + m(S).
\end{aligned}$$

Now, $0 \leq 2k - 2 - l \leq k - 1$ so

$$max(Ap(S)) - x \in (3n-6)(2k-1) + [0, k-1] + m(S).$$

The analysis of T_1 demonstrated that

$$(3n-6)(2k-1) + [0, k-1] + m(S) \subseteq Ap(S).$$

3. Other $k \times n$ Pefect Bricks

We conjecture that the infinite family of perfect bricks defined in (1.2) is, in fact, just one family in an infinite collection of such families. Indeed, let k, n, and j be integers with $k, n \geq 2$ and $j \geq 1$. Define

$$m(k,n,j) = (3j^2+6j+1)(k-2) + (n-2)((3j^2+3j)(k-2) + (3j^2+5j+1)) + (3j^2+8j+4),$$

and S to be the numerical semigroup with generating set

$$\bigcup_{i=n}^{2n-1} m(k,i,j) + [0, k-1].$$

We conjecture that S admits a $k \times n$ perfect brick with

$I = (0, 1, \ldots, k-1)$ and $S - I = (m(k,n,j), m(k,n+1,j), \ldots, m(k,2n-1,j))$.

We further conjecture that all such numerical semigroups are symmetric.

(3.1) Example: Let $k = 3$, $n = 5$ and $j = 3$. Then

$S = < 338, 339, 340, 417, 418, 419, 496, 497, 498, 575, 576, 577, 654, 655, 656 >$.

If we define $I = (0, 1, 2)$, it can be verified that (S, I) forms a 3×5 perfect brick.

The family defined in (1.2) results by letting $j = 1$. Investigations of the family that results for $j = 2$ indicate that an analysis similar to the one in Section 2 will confirm that all such numerical semigroups admit $k \times n$ perfect bricks and are symmetric. An attempt to conduct such an analysis for all values of j proved too complex to manage but checks of many of the semigroups in these families support our conjecture.

In addition to these (conjectured) families of perfect bricks, the following family is offered by Jeff Rushall:

Let k and n be integers with $k \geq 2$ and $n \geq 3$. Define

$$m(k, n) = 10(k-2) + (n-3)(3(k-2)+5) + 17,$$

and S to be the numerical semigroup with minimal generating set

$$\bigcup_{i=n}^{2n-1} m(k, i) + [0, k-1].$$

Using an analysis similar to that given in Section 2, one can show that every semigroup in this family is non-symmetric and admits a $k \times n$ perfect brick with

$$I = (0, 1, \ldots, k-1) \text{ and } S - I = (m(k,n), m(k,n+1), \ldots, m(k, 2n-1)).$$

(3.2) Example: Let $k = 3$ and $n = 4$. Then

$$S =< 35, 36, 37, 43, 44, 45, 51, 52, 53, 59, 60, 61 > .$$

If we define $I = (0, 1, 2)$, then (S, I) forms a 3×4 perfect brick.

Concluding Remarks

This investigation answers a question from [7]. In that paper, the notions of balanced and unitary are introduced for numerical semigroups of embedding dimension 4, and it is shown that such semigroups are unitary if and only if they admit a 2×2 perfect brick. At the end of that paper, it is mentioned that it would be worthwhile to investigate if these notions can be generalized to $k \times n$ bricks. Throughout our research into the families discovered here we looked for such generalizations, but without any real progress. However, we still believe that such ideas are worth pursuing.

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